INTERNATIONAL MATHEMATICS
TOURNAMENT OF TOWNS

SENIOR PAPER: YEARS 11,12

Tournament 42, Northern Spring 2021 (A Level)

Note: Each contestant is credited with the largest sum of points obtained for three problems.

1. In a room there are several children and a pile of 1000 sweets. The children come to the pile one after another in some order. Upon reaching the pile every child divides the current number of sweets in the pile by the number of children currently in the room, rounds the result if it is not integer, takes the resulting number of sweets from the pile and leaves the room. All the boys round upwards and all the girls round downwards. The process continues until everyone leaves the room. Prove that the total number of sweets received by the boys does not depend on the order in which the children approach the pile. (4 points)

2. Does there exist a positive integer n such that for all real x and y there exist real numbers $a_1, \ldots, a_n$ satisfying

$$x = a_1 + \ldots + a_n \quad \text{and} \quad y = \frac{1}{a_1} + \ldots + \frac{1}{a_n} \ ?$$

(5 points)

3. Let M be the midpoint of the side BC of the triangle ABC. The circle ω passes through A, touches the line BC at M, intersects the side AB at the point D and the side AC at the point E. Let X and Y be the midpoints of BE and CD respectively. Prove that the circumcircle of the triangle MXY touches ω. (5 points)

4. There is a row of $100N$ ham sandwiches. A boy and his cat play a game. In one *action* the boy eats the first sandwich from any end of the row. In one *action* the cat either eats the ham from one sandwich or does nothing. The boy performs 100 actions in each of his turns, and the cat makes only 1 action per turn. The boy starts first and then he and the cat take turns. The boy wins if the last sandwich he eats contains ham. Is it true that he can win for any positive integer N no matter how the cat plays? (8 points)

5. 100 tourists arrive at a hotel at night. They know that in the hotel there are single rooms numbered $1, 2, \ldots, n$, and among them k rooms (the tourists do not know their numbers) are under repair, the other rooms are vacant. The tourists, one after another, check the rooms in any order (maybe different for different tourists), and the first room not under repair is taken by the tourist. Tourists do not see which rooms are occupied by others but they do not want to bother their peers and check already occupied rooms. Therefore they coordinate their strategy beforehand to avoid this situation. For each k find the smallest n for which the tourists may occupy their rooms for sure. (8 points)

6. Find at least one real number A such that for any positive integer n the distance between $\lceil A^n \rceil$ and the nearest square of an integer is equal to 2. (By $\lceil x \rceil$ we denote the smallest integer not less than x.) (10 points)

7. An integer $n > 2$ is given. Peter wants to draw n arcs of length α and radius 1 on a unit sphere (i.e. each arc is a part of a great circle) so that they do not intersect each other. Prove that

(a) for all $\alpha < \pi + \frac{2\pi}{n}$ it is possible; (6 points)

(b) for all $\alpha > \pi + \frac{2\pi}{n}$ it is impossible. (7 points)